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Measurements in rotating systems

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Abstract. The paper considers the application of the metric

$$ds^2 = dr^2 + r^2\,d\theta^2 + dz^2 + 2\omega r^2\,d\theta\,dt - (c^2 - \omega^2 r^2)\,dt^2$$

to a rotating system. This metric has been used by other investigators but the interpretation of measurements made within the rotating system through the framework of the metric has never been properly performed.

The first part of the paper considers the application of radar measurement to the rotating system. In the second part infinitesimal radar measurements are used to show that for an observer at a radius R within a rotating system, similar results are obtained, through the application of the metric, to those which would be obtained by the use of instantaneous rest frames. An error in a paper by Grøn is pointed out.

The overall aim of this paper is to clarify the interpretation of measurements made by a rotating observer when the metric approach to rotating systems is used.

1. Introduction

The cylindrical polar form of the Minkowski metric in the system $\bar{S}(\bar{r}, \bar{\theta}, \bar{z}, \bar{t})$ is

$$ds^2 = d\bar{r}^2 + \bar{r}^2\,d\bar{\theta}^2 + d\bar{z}^2 - c^2\,d\bar{t}^2$$

and the Galilean rotational transformation is

$$r = \bar{r}, \qquad \theta = \bar{\theta} - \omega\bar{t}, \qquad z = \bar{z}, \qquad t = \bar{t}.$$

From these two equation we can obtain in the system $S(r, \theta, z, t)$

$$ds^2 = dr^2 + r^2\,d\theta^2 + dz^2 + 2\omega r^2\,d\theta\,dt - (c^2 - \omega^2 r^2)\,dt^2. \tag{1}$$

In this paper an investigation of this metric is presented. The metric describes a particular rotating system obtained by a Galilean rotation of a cylindrical polar coordinate system about its own z axis. Hence, this is the space–time of the rotating system as seen by an observer situated at the origin of coordinates, O, and who is rotating with the system. The Riemann–Christoffel and Ricci tensors obtained from the metric coefficients in equation (1) are, of course, equal to zero, and so the metric corresponds to a flat space–time. This is an important point for, unlike the metrics derived by Kursunoglu (1951) and Mikhail and Abdalla (1965) the metric corresponds to a space–time in which there is zero energy distribution.

This metric has often been used to describe rotating systems but the correct interpretation, through the metric, of observations made by an observer who is rotating about some point, seems to be lacking. It is the intention of this paper to clarify this point.

2. Measurements in rotating systems: fundamentals

Let the metric in equation (1) have origin of coordinates at O. How can we relate the measurements made of some event by an observer at O to measurements of the same event made by an observer at O′ where O′ is situated at $r=R$, $\theta=0$, $z=0$?

We cannot use a linear transformation because the units with which an observer at O′ makes measurements may be different from those with which an observer at O makes measurements. It can easily be shown that the proper clocks at O and at O′ run at different rates since from equation (1) we have at O(0, 0, 0, t)

$$ds^2=-c^2\,dt^2$$

and at O′(R, 0, 0, t)

$$ds^2=-(c^2-\omega^2R^2)\,dt^2.$$

If we take $ds^2=-c^2\,d\tau^2$, where τ is the proper time of a clock at O′, we have

$$d\tau=\pm(1-\omega^2R^2/c^2)^{1/2}\,dt. \tag{2}$$

The positive root of this equation is usually considered applicable.

We do not yet have the transformation involving the spatial coordinates of observers at O and at O′. But if we are to discuss any 'transformation of coordinates' we must know exactly what the coordinates represent before we can relate any physical meaning to the transformation.

In special relativity the spatial coordinates of a Lorentz frame correspond exactly to quantities measured by radar distance, luminosity distance, apparent size and triangulation measurements. In the system given by equation (1) it is not obvious that these various methods of measurement must give the same result. For convenience we choose radar distance as the form of measurement to be used in the rotating system under investigation.

To be more specific we define exactly what is meant by radar measurement. We assume that an observer, who always measures the 'two-way' velocity of light to be constant and equal to c, emits a radar signal at τ_1 on his local (proper) clock. The radar signal is reflected from some body and is then received by the observer at τ_2. The radar distance of the body from the observer, as measured by the observer, is then defined as

$$r_d=\tfrac{1}{2}c(\tau_2-\tau_1) \tag{3}$$

Note the assumption that the 'two-way' velocity of light is constant. The velocity of light in one direction need not necessarily be constant.

Consider now the direction of the body as seen by the observer. It is quite possible that the emitted and reflected radar rays will be emitted and returned in different directions as seen by an observer at O. As a convention we take the direction of the body to be along a line bisecting the angle between the emitted and received radar rays. Justification for this is, as will be seen below, that these radar coordinates as defined, correspond exactly to the coordinate system S used by an observer at the origin of the coordinates, O.

We now need to know the equation describing the paths of light rays in the system S, and these are, of course, the null-geodesics of the metric of equation (1).

Rewriting equation (1) as

$$(ds/d\lambda)^2=\dot r^2+r^2\dot\theta^2+\dot z^2+2\omega r^2\dot\theta\dot t-(c^2-\omega^2r^2)\dot t^2=g_{\mu\nu}\dot x^\mu\dot x^\nu \tag{4}$$

where λ is a parameter which varies along the curve and $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$, we can find the null-geodesic equations from the Euler–Lagrange variational equations;

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial}{\partial \dot{x}^\mu}(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)\right) = \frac{\partial}{\partial x^\mu}(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu) \tag{5}$$

and the additional condition:

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \tag{6}$$

Applying these equations to equation (4) we obtain

$$(\mathrm{d}/\mathrm{d}\lambda)(2\dot{r}) = 2r\dot{\theta}^2 + 4\omega r\dot{\theta}\dot{t} + 2\omega^2 r\dot{t}^2 \tag{7}$$

$$(\mathrm{d}/\mathrm{d}\lambda)(2r^2\dot{\theta} + 2\omega r^2\dot{t}) = 0 \tag{8}$$

$$(\mathrm{d}/\mathrm{d}\lambda)(2\dot{z}) = 0 \tag{9}$$

$$(\mathrm{d}/\mathrm{d}\lambda)[2\omega r^2\dot{\theta} - 2\dot{t}(c^2 - r^2\omega^2)] = 0 \tag{10}$$

and

$$\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 + 2\omega r^2\dot{\theta}\dot{t} - (c^2 - \omega^2 r^2)\dot{t}^2 = 0. \tag{11}$$

Integrating equations (8) to (10) produces

$$r^2\dot{\theta} + \omega r^2\dot{t} = A \tag{12}$$

$$\dot{z} = B \tag{13}$$

$$\omega r^2\dot{\theta} - \dot{t}(c^2 - r^2\omega^2) = D \tag{14}$$

from which comes

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{Br^2c^2}{Ac^2 - A\omega^2 r + \omega r^2 D} \tag{15}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{Bc^2}{A\omega - D} \tag{16}$$

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{r^2(A\omega - D)}{Ac^2 - A\omega^2 r^2 + \omega r^2 D} \tag{17}$$

where A, B and D are constants of integration. Multiplying through by $(1/\dot{\theta})^2$ in equation (11) gives

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + 2\omega r^2\left(\frac{\mathrm{d}t}{\mathrm{d}\theta}\right) - (c^2 - \omega^2 r^2)\left(\frac{\mathrm{d}t}{\mathrm{d}\theta}\right)^2 = 0. \tag{18}$$

Substituting from equations (15) to (17) in this equation and using $B = 0$, yields

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r[(A^2\omega^2c^2 + D^2c^2 - 2A\omega Dc^2)r^2 - A^2c^4]^{1/2}}{Ac^2 - A\omega^2 r^2 + \omega r^2 D} \tag{19}$$

If we now let $r = a$ when the ray is at its point of closest approach to the origin, then at a, $(\mathrm{d}r/\mathrm{d}\theta) = 0$ and so from equation (19) we have

$$a = \pm Ac/(A\omega - D).$$

Substituting this into equations (15) to (17) and rearranging gives

$$\frac{\mathrm{d}\theta}{\mathrm{d}r}=\frac{\pm a}{r(r^2-a^2)^{1/2}}\pm\frac{\omega}{c}\frac{r}{(r^2-a^2)^{1/2}} \tag{20}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\pm\frac{ac}{r^2}\pm\omega \tag{21}$$

and from these equations we can obtain

$$\frac{\mathrm{d}t}{\mathrm{d}r}=\frac{\pm r}{c(r^2-a^2)^{1/2}}. \tag{22}$$

Integration of equations (20) and (21) produces

$$\theta=\pm\cos^{-1}(a/r)\pm(\omega/c)(r^2-a^2)^{1/2}+k_1 \tag{23}$$

$$t=\pm(1/c)(r^2-a^2)^{1/2}+k_2 \tag{24}$$

where k_1 and k_2 are constants of integration ($k_1=\theta$, $k_2=t$, at $r=a$).

These are the equations for the null-geodesics, the paths of light rays, in the system described by the metric of equation (1).

We now examine the relationship between the radar coordinates and the coordinates already in use by an observer at O, which are, of course, r, θ, z and t. The radial velocity of rays emitted from O and received by O will be given by equation (22) with $a=0$ and is

$$\mathrm{d}r/\mathrm{d}t=\pm c \tag{25}$$

So the radar coordinate $r_d=\frac{1}{2}c(t_2-t_1)$ is the same as the radial coordinate already in use.

Also from equation (23) with $a=0$ we have

$$\theta=\pm(\omega r/c)+k_1\pm\pi/2. \tag{26}$$

Hence the line bisecting the direction of the emitted and reflected radar rays points along the same value of θ as given by the angular coordinate already used.

We can now return to the problem of relating measurements by observers at O and at O′.

Consider figure 1. An observer at O cannot himself make a measurement of the distance of P from O′. But he can locate the positions of P and O′, and then the 'implied' distance between the two points is given by the cosine rule, in which case we write

$$\mathrm{O'P_{(O)}}=(R^2+r_1^2-2r_1R\cos\theta_1)^{1/2}. \tag{27}$$

To find the radar distance $\mathrm{O'P_{(O')}}$ as measured by O′ it is necessary to find the time of flight of a photon from O′ to P and the time of flight of another photon from P to O′. Converting these times to the times on the proper clock of O′ by means of equation (2), adding them together and multiplying by $c/2$, we find the radar distance measured by O′. For a constant angular velocity there are only two null-geodesics which join O′ and P. One corresponds to the track of a photon from O′ to P and the other corresponds to the track of a photon from P to O′. The two paths may, of course, be different.

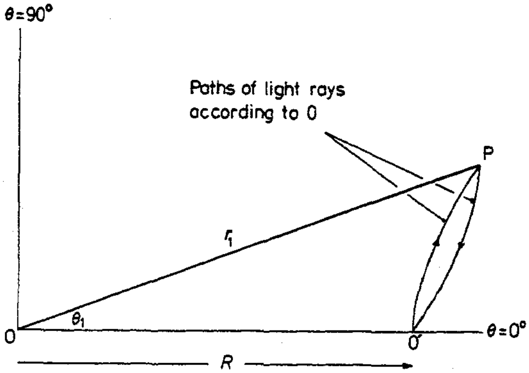

Figure 1. Paths of light rays from O′ to P and from P to O′, according to the coordinate system S, in use by an observer at O.

Consider first the emitted ray. Let the ray be at $r = a$ at $t = 0$, at O′ at $t = t_0$ and at P at $t = t_1$; then from equations (23) and (24) we have,

$$O = \pm\cos^{-1}(a/R) \pm (\omega/c)(R^2 - a^2)^{1/2} + k_1 \tag{28}$$

$$\theta_1 = \pm\cos^{-1}(a/r_1) \pm (\omega/c)(r_1^2 - a^2)^{1/2} + k_1 \tag{29}$$

$$t_1 = \pm(1/c)(r_1^2 - a^2)^{1/2} \tag{30}$$

$$t_0 = \pm(1/c)(R^2 - a^2)^{1/2}. \tag{31}$$

It is now necessary to solve equations (28) and (29) for k_1 and, in particular, a. Substitution into (30) and (31) would then yield $(t_1 - t_0)$, the time of flight of a photon from O′ to P, as calculated by an observer at O. Up to present it has proved impossible to solve equations (28) and (29) for k_1 and a, although it may be possible to obtain an approximate solution by computer.

The only case so far where it has proved possible to relate integrated radar distance measurements of observers at O and at O′ is the obvious case of the radar distance measurement of O′ from O by an observer at O, and the radar distance measurement of O from O′ by an observer at O′.

The radar distance measurement of O′ from O by an observer at O, is given by

$$r_{d(O)} = R$$

and the radar distance measurement of O from O′ by an observer at O′ is

$$r_{d(O')} = \tfrac{1}{2}c(\tau_2 - \tau_1)$$

where τ_1, τ_2 are the times of departure and arrival respectively, of a radar ray reflected from O.

With these equations and equation (2) we may obtain

$$r_{d(O')} = R(1 - \omega^2 R^2/c^2)^{1/2}. \tag{32}$$

This equation has previously been derived by Jennison (1964).

Note also that if ω is the angular velocity of O′ about the $\bar{z}$ axis of the system $\bar{S}$ and ω' is the angular velocity of the system $\bar{S}$ about O′ as measured by an observer at O′ by the timing of successive transits of some point fixed in the system $\bar{S}$, then because of the time dilatation given by equation (2) we have

$$\omega' = \omega/(1-\omega^2 R^2/c^2)^{1/2}. \tag{33}$$

This equation, which was also produced by Jennison (1964), has been discussed by Davies and Jennison (1975).

It is possible to find a relationship between infinitesimal radar distance measurements of observers at O and at O′ which is valid for a vanishingly small region around O′. Thus we can at least relate the angles of incidence of a light ray as seen by an observer at O′ to that calculated by an observer at O.

3. Elemental radar distance measurement

We again use figure 1, however we now let P be at some infinitesimal distance from O′. As before, we allow the observer at O′ to emit a radar signal which is reflected at P. The world lines of O′, P and the radar rays, relative to an observer at O, are drawn in figure 2. Note that P may have relative motion with respect to an observer at O′.

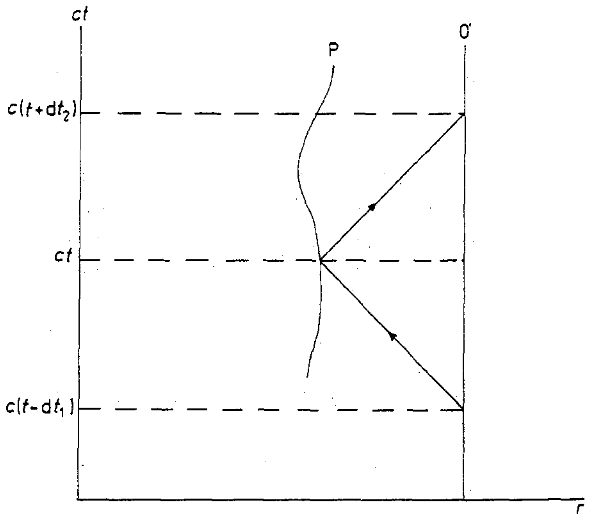

Figure 2. The world lines of light rays from O′ to P and from P to O′.

The path of the light ray is given by $ds = 0$ and so from equation (1) we have

$$0 = dr^2 + r^2\,d\theta^2 + dz^2 + 2\omega r^2\,d\theta\,dt - (c^2 - \omega^2 r^2)\,dt^2. \tag{34}$$

Solving this equation as a quadratic in terms of dt we obtain

$$dt = \frac{\omega r^2\,d\theta \mp [\omega^2 r^4\,d\theta^2 + (c^2-\omega^2 r^2)(dr^2 + r^2\,d\theta^2 + dz^2)]^{1/2}}{c^2 - \omega^2 r^2}. \tag{35}$$

The two roots correspond, as may be seen from figure 2, to propagation in the directions O′P and PO′. Taking the difference of the two roots we find the time for a ray to travel from O′ to P and back, which is

$$dt_2 - dt_1 = \frac{2[\omega^2 r^4\, d\theta^2 + (c^2 - \omega^2 r^4)(dr^2 + r^2\, d\theta^2 + dz^2)]^{1/2}}{c^2 - \omega^2 r^2}. \tag{36}$$

The infinitesimal radar distance $d\sigma$, measured by an observer at O′, we define as

$$d\sigma = \tfrac{1}{2} c(d\tau_2 - d\tau_1). \tag{37}$$

From this equation and equations (2) and (36) we produce

$$d\sigma^2 = dr^2 + \frac{r^2\, d\theta^2}{1 - \omega^2 r^2/c^2} + dz^2 \tag{38}$$

which since O′ is at $(R, 0, 0, t)$ in the system S, may be written

$$d\sigma^2 = dr^2 + \frac{R^2\, d\theta^2}{1 - \omega^2 R^2/c^2} + dz^2. \tag{39}$$

Note that equation (38) gives the radar measure for an elemental area around O′ actually measured by an observer at O′ but expressed in the coordinates of an observer at O. So in a direction normal to the radius and in the plane of rotation, when an observer at O measures an elemental distance $r\, d\theta$ at observer at O′ measures an elemental distance $r\, d\theta/(1 - \omega^2 r^2/c^2)^{1/2}$. We might also notice the difference in the expression for radial distance measurements when $d\theta = dz = 0$ in equation (38) which gives $d\sigma = dr$, in contrast to the relation between integrated radar distance measurements as in equation (32).

It should also be noted that in deriving equation (38) we have assumed for a local observer that the 'two-way' velocity of light is constant and equal to c. By this we mean that any local measurement of the velocity of light over opposite directions by experiments such as that by Michelson (1927), must always produce a constant value equal to c, for the velocity of light. This does not, of course, mean that the velocity of light in one direction need necessarily be invariant. However, since all experiments on the velocity of light to date have involved two-way measurements, we might consider the assumption as discussed above to be a reasonable one since it is in line with the principle of special relativity.

It may be relevant at this point to discuss the one-way velocity of light in the radial and the tangential directions as seen by an observer at O′, since in a recent publication (Grøn 1975) the expressions for the local velocities of light in these directions were, I believe, incorrectly given.

For a ray travelling radially at O′ we have from equation (1)

$$0 = dr^2 - (c^2 - \omega^2 r^2)\, dt^2. \tag{40}$$

This equation together with equation (2) gives us

$$dr/d\tau = \pm c \tag{41}$$

and hence in both the positive and negative radial directions the velocity of light at O′, measured by an observer at O′, is constant and equal to c.

For a ray travelling tangentially at O′, from equation (1) we have

$$0 = r^2\, d\theta^2 + 2\omega r^2\, d\theta\, dt - (c^2 - \omega^2 r^2)\, dt^2. \tag{42}$$

Solving this as a quadratic for $r\,\mathrm{d}\theta/\mathrm{d}t$ we find

$$r\,\mathrm{d}\theta/\mathrm{d}t = -\omega r \pm c. \tag{43}$$

Now from equation (38) we have for an observer at O′, for this ray

$$\mathrm{d}\sigma^2 = r^2\,\mathrm{d}\theta^2/(1-\omega^2 r^2/c^2). \tag{44}$$

Using this equation and equations (2) and (43) we can obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \frac{-\omega r \pm c}{1-\omega^2 r^2/c^2} \tag{45}$$

for a ray travelling tangentially at O′.

Equations (41) and (45) are in contrast to those obtained by Grøn (1975) who claimed, in equations (47) and (50) of his paper, that the local radial velocity of light is

$$c_{\mathrm{r}} = (1-\omega^2 r^2/c^2)^{1/2} c \tag{46}$$

and the local tangential velocity of light is

$$c_{\mathrm{t}} = (1-\omega^2 r^2/c^2)^{-1/2}(c-\omega r). \tag{47}$$

The mistake in the equations for the velocities presented by Grøn is obvious. He has, in fact, calculated a coordinate velocity of light in a system using the spatial coordinates of an observer at O′, and the temporal coordinate of an observer at O, rather than calculating the local velocity for an observer at O′.

We can check that the 'two-way' velocity of light is constant and equal to c by the following method. We find the element $\mathrm{d}\sigma$ for a ray travelling from O′ to P and the element for a ray travelling from P to O′, and add. If the 'two-way' velocity of light is constant and equal to c then the addition should produce $2\,\mathrm{d}\sigma$. Since the velocity in the radial direction has been shown to be $\pm c$ it is only necessary to follow this procedure for the tangential rays. We let $(\mathrm{d}\sigma/\mathrm{d}\tau)_{1,2}$ and $\mathrm{d}\tau_{1,2}$ be the velocities and times of travel for a ray in the two opposite tangential directions respectively, then the total distance travelled by the rays, according to an observer at O′ is

$$X = (\mathrm{d}\sigma/\mathrm{d}\tau)_1\,\mathrm{d}\tau_1 + (\mathrm{d}\sigma/\mathrm{d}\tau)_2\,\mathrm{d}\tau_2. \tag{48}$$

Using equations (2), (35) and (45) in this equation produces

$$X = 2\,\mathrm{d}\sigma = 2r\,\mathrm{d}\theta/(1-\omega^2 r^2/c^2)^{1/2} \tag{49}$$

as indeed it must, since we have assumed that the 'two-way' velocity of light is constant in deriving $\mathrm{d}\sigma$.

Note that it is not possible merely to take the average of the two velocities in equation (45) to obtain the one-way velocity of light since the rays travel along different paths.

Let us now once again consider a photon which has travelled from O to O′ but let us examine the path of the photon as seen by an observer at O′, when the photon is very near to O′. The situation, as seen by an observer at O, is drawn in figure 3.

We let an angle ϕ' be the angle between the path of the light beam, and the direction of O from O′, as seen by an observer at O′.

When the photon is near to O′ an observer at O will measure that the photon is a distance $\mathrm{d}r$ from O′ along a line joining O and O′ and at a distance $r\,\mathrm{d}\theta$ along a line at right angles to this line, in the plane of rotation. However, an observer at O′ will

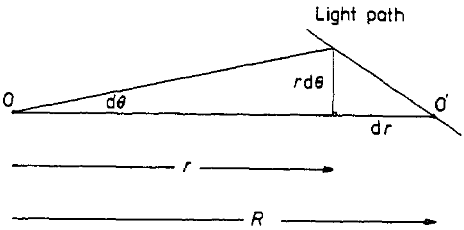

Figure 3. The light path near to O′ as seen by an observer at O using coordinates of the system S.

measure that the photon is a distance dr along the radial line towards O and $r\,d\theta/(1-\omega^2r^2/c^2)^{1/2}$ along the line normal to the radius, if the measurements are made close enough to O′ for the assumption $r \simeq R$ to be valid. The expressions above are only valid for distances measured in an elemental area around O′.

Applying these arguments to the equation (20) for the null-geodesics, we find that the equation for the null-geodesics near to O′, as seen by an observer at O′, is given by

$$\left(\frac{d\theta}{dr}\right)' = \frac{\pm(ac \pm \omega r^2)}{rc[(r^2-a^2)(1-\omega^2r^2/c^2)]^{1/2}} \tag{50}$$

or

$$\left(\frac{d\theta}{dr}\right)' \simeq \frac{\pm(ac \pm \omega R^2)}{Rc[(R^2-a^2)(1-\omega^2R^2/c^2)]^{1/2}}. \tag{51}$$

From the definition of ϕ' and from figure 3 we have

$$\tan\phi' = (r\,d\theta/dt)'. \tag{52}$$

From this and equation (51) we produce

$$\tan\phi' = \frac{\pm(ac \pm \omega R^2)}{c[(R^2-a^2)(1-\omega^2R^2/c^2)]^{1/2}}. \tag{53}$$

For a ray which travels through O and hence has $a=0$, this equation reduces to

$$\tan\phi = \frac{\pm\omega R}{c(1-\omega^2R^2/c^2)^{1/2}}. \tag{54}$$

This is precisely the same equation as would be obtained by associating an instantaneous Lorentz frame with O′ and calculating the angle of the incoming ray initially emitted by O.

We should note that in this analysis equations (50) and (51) are only considered valid for any small area close to O′. However, this is sufficient to calculate the angle of incidence at O of photons arriving from, and departing to, O.

From equation (54) and the restriction on the constancy of the local velocity of light we may calculate the radial and tangential two-way velocities of light rays at O′ as measured by an observer at O′. These will, of course, agree with the values calculated by using the method of instantaneous Lorentz transformations.

4. Summary

This paper has considered a method of making measurements within a particular metric framework for a rotating system. The relationship between measurements made by observers at different points within a rotating system has been investigated. Only the relationships between elemental measurements for an area near to a rotating observer are so far available, but thèse relationships have been shown to agree with results which would be produced by the application of instantaneous Lorentz frames to rotation.

The agreement depends on the assumption of a locally constant 'two-way' velocity of light. It is not necessary to assume that the local velocity of light in one direction is constant.

Many of the problems of rotation in relativity theory are involved with the method of making measurements within the rotating system and with the assumptions made in deriving results through a particular method of measurement. It is hoped that this paper has clarified some of the relationships between observers at different points in a rotating system.

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